Solution sketch 2 - Computational Models - Spring 2014

1. (a) L_1 is not regular. Can be proved by the pumping lemma. For any p choose $w = a^{(p+3)!}$. Thus $w \in L_1$ and $|w| \geq p$. Let $w = xyz$ s.t. $k = |y| > 0$ $|xy| < p$. For $i = 2$, $xy^iz = a^{(p+3)!+k} \notin L_1$ since $(p+3)! < (p+3)!+k \leq (p+3)!+p < (p+3)!+(p+3)! = 2(p+3)! < (p+4)!$

   (b) L_2 is not regular. Can be proved by the pumping lemma. For any p choose $s = 1^p01^p01^{2p+2}$. Thus $w \in L_2$ and $|w| \geq p$. Let $s = xyz$ s.t. $k = |y| > 0$ $|xy| \leq p$. Choose $i = 2$, $y = 1^k$ and $xy^2z = 1^{p+k}01^p01^{2p+2}$. If k is odd $|xy^2z|$ is odd, and thus cannot be a concatenation of uvw where $|uv| = |w|$. Otherwise the first half of s is $1^{p+k}01^{p-\frac{k}{2}+1}$ and since it contains only one 0 it can't be written as uv s.t. $\#_0(u) = \#_0(v)$.

   (c) L_3 is not regular. Can be proved by the pumping lemma. For any p choose $w = (101)^p(010)^p$. $\#_{010}(w) = \#_{101}(w) = p + 1$ thus $w \in L_3$. Let $w = xyz$ s.t. $|y| > 0$ $|xy| < p$. For $i = 0$ $xy^iz \notin L$. y is a substring of $(101)^p$. 2 possible cases (1) $\#_0(y) \geq 1$ and (2) $y \in \{1, 11\}$. Show that for each case $xy^0z \notin L$

2. (a) Can be proved by Myhill-Nerode Theorem. Note that for every $m_1 \neq m_2$, $b^{m_1} \not\sim_{L_1} b^{m_2}$.

   (b) Define a homomorphism h as $h(a) = 0,\ h(b) = 0,\ h(c) = 1$. $h(L_2) = \{a^nb^b \mid n > 0\}$

   (c) Define a homomorphism h_1 as $h_1(a) = a,\ h_1(b) = b,\ h_1(c) = a$.

   - $L_3' = h^{-1}(L_3) = \{(a \cup c)^n b(a \cup c)^n \mid n \geq 0\}$
   - $L_3'' = L_3' \cap a^*bc^* = \{a^nbc^n \mid n \geq 3\}$

   Define a homomorphism h_2 as $h_2(a) = 1,\ h_2(b) = \varepsilon,\ h_2(c) = 0$. $h_2(L_3'') = \{1^n0^n \mid n \geq 0\}$

3. (a) L_1 is context free. $L_1 = \mathcal{L}(\langle\{S\}, \{a, b\}, \{S \to aaSbbb, S \to \epsilon\}, S\rangle)$.

(b) L_2 is not context free. Proof by the pumping lemma. For any p choose $w = 0^m$ s.t. m is prime and $m \geq p$. Thus $w \in L_2$ and $|w| \geq p$. Let $w = uvxyz$ s.t. $|vy| > 0$ and $|vxy| \leq p$. We denote $|v| = k$ and $|y| = l$. For $i = m+1$, $uv^ixy^iz = 0^{m+k\cdot m+l\cdot m} \notin L_2$ since $m + k \cdot m + l \cdot m = m \cdot (1 + k + l)$ is not prime.

(c) L_3 is not context free. Proof by the pumping lemma. For any p choose $w = a^pb^pca^pb^p$. Thus $w \in L$ and $|w| \geq p$. Let $w = uvxyz$ s.t. $|vy| > 0$ and $|vxy| \leq p$.

- If y or v contains c, $uv^0xy^0z = uxz \in L_3$ since there is no c in it.
- Otherwise, 3 possible cases:
  i. vxy is in the first part of w (before the c), therefore $xyz = a^ib^i$. Thus $uv^2xy^2z \notin L_3$ because the first part in longer than the second part.
  ii. vxy is in the second part of w (after the c), therefore $xyz = a^ib^i$. Thus $uv^0xy^0z \notin L_3$ because the first part in longer than the second part.
  iii. If vxy is in the middle of w, i.e. $vxy = b^ica^j$
    - If $j > 0$, then $uv^0xy^0z \notin L_3$ (less a in the second part)
    - If $i > 0$, then $uv^2xy^2z \notin L_3$ (more b in the first part)

(d) $L_4 = \mathcal{L}(\langle\{S\}, \{0,1\}, \{S \to 0S0|1S1|0X1|1X0, X \to 0X|1X|\epsilon\}, S\rangle)$. Prove that $w \in L_4$ iff is of the form $w = uav\bar{a}u^R$ where X derives $v \in \{0,1\}$ and S derives $uaX\bar{a}u^R$.

4. (a) Pump $w = 0^n1^n0^n1^n0^n1^n$

(b)

$$S \to XX|YY|X|Y$$

$$X \to 0X0|0X1|1X0|1X1|1$$

$$Y \to 0Y0|0Y1|1Y0|1Y1|0$$

It implies that the context-free languages are not closed under complementation.

5. Let $L_1 = \{0^n1^n \mid n \geq 0\}$ and $L_2 = \{2\}$.
$L = L_1L_2L_1 = \{0^n1^n20^m1^m \mid n, m \geq 0\}$ is context free since both L_1 and L_2 are.
Let $L' = DropMiddle(L) \cap (0 \cup 1)^* = \{0^n1^n0^n1^n \mid n \geq 0\}$, but L' is not context free. Proof by the pumping lemma with $w = 0^p1^p0^p1^p$

6. (a) **Claim:** Let A be a DFA with n states. $|L(A)|$ is infinite iff $\exists w \in L(A),\ n < |w| \leq 2n$.
Proof: If $\exists w.|w| > n$ then as we learned in the pumping lemma, this word can be pumped infinitely and therefore $L(A)$ is infinite. if $L(A)$ is infinite, then there is a word $w \in L(A)$ such that $|w| > n$. The run of this word in A contains a cycle. We remove all cycles from the run and remember one simple cycle c, $|c| \leq n$. The run without the cycles give a word $w' \in L(A), |w'| < n$. We start pumping w' with the cycle c and we will eventually get a word in $L(A)$ in the proper length.
This means that given a DFA A, we can run in A all the words w, such that $n < |w| \leq 2n$. If one of the words is accepted, then $L(A)$ is infinite, otherwise - finite.

(b) First we check if $L(A)$ is infinite. If it is, we return $false$. otherwise, we run in A all words of length at most n and count how many are accepted. we return $true$ iff the count is 9,122,009.

(c) Note that
$L(A_1) = L(A_2) \Leftrightarrow L(A_1) \subseteq L(A_2) \wedge L(A_2) \subseteq L(A_1) \Leftrightarrow$
$L(A_1) \setminus L(A_2) = \emptyset \wedge L(A_2) \setminus L(A_1) = \emptyset \Leftrightarrow$
$(L(A_1) \setminus L(A_2)) \cup (L(A_2) \setminus L(A_1)) = \emptyset$.
To check if $L(A_1) = L(A_2)$ construct the DFA D for $(L(A_1) \setminus L(A_2)) \cup (L(A_2) \setminus L(A_1))$ and check if $L(D)$ is empty (how?).

(d) **Claim:** Let G be a CFG with n variables. $|L(A)|$ is infinite iff $\exists w \in L(A),\ 2^n < |w| \leq 2^{n+1}$.
Proof: If $\exists w.|w| > 2^n$ then as we learned in the pumping lemma, this word can be pumped infinitely and therefore $L(G)$ is infinite. If $L(G)$ is infinite then there is a word $w \in L(G)$ such that $|w| > 2^n$. The minimum size parse tree for this word contains a variable A that appears twice on some path from the root to some leaf. We will remove all duplicate variables on the same path by shrinking as we learned in the pumping lemma, and get a word w', $|w'| \leq 2^n$. Then we start pumping A. Assuming we chose A to be the lowest variable that appear twice on some path, it is guaranteed that $|vy| \leq 2^n$ and we will eventually get a word in $L(G)$ of the proper length.
This means that given a CFG G we can check if G generates any word w, such that $2^n < |w| \leq 2^{n+1}$. If one of the words is generated by G, then $L(G)$ is inifinite, otherwise - finite.