Computational Models, Spring 2014 Exrecise #3

Turing Machines

1. Give a verbal description of a Turing Machine that accepts the following languages. There is no need to give a formal definition but you should give a detailed explanation of each step.

   (a) $L = \{w \in \Sigma^* \mid \#_a(w) = \#_b(w) = \#_c(w)\}$ above $\Sigma = \{a, b, c\}$

   (b) $L = \{\{0\}^{n^2} \mid n \in N \text{ and } n > 0\}$ above $\Sigma = \{0, 1\}$

   (c) $L = \{a^i b^j c^k \mid i \cdot j = k \text{ and } i, j, k \geq 0\}$ above $\Sigma = \{a, b, c\}$

2. In this question we are interested in *computing a function* using Turing Machines.

   (a) Given the function $f(x) = x + 1$, construct a TM that given a number x, represented in binary form on the tape where the least significant bit (lsb) is on the left-most position, returns the value $f(x)$. Give a *formal* description. There is no need to prove your answer.

   (b) How would your answer to (a) change if the number was represented when the most significant bit (msb) is on the left-most position? No need to formally build the TM, explain in pseudo-code your algorithm.

   (c) For the following question, assume that the Turing machine has a an *input* tape where the input is written (it is read only) and an *output tape* where the output is written. Describe a Turing Machine such that given an input x (in Unary representation) returns the value $\lfloor \log_2(x) \rfloor$. No need to formally build the TM, explain in pseudo-code your algorithm and explain its correctness.

3. Let us define a generalization of Turing Machines to include a *finite memory* of size n. We denote such a Turing Machine formally as:

   $$M_{\text{mem}} = (Q, \Sigma, \gamma, \delta_{\text{mem}}, n, q_0, qa, q_r),$$

   where all the definitions are identical to the Turing Machine defined in class except that there is the *finite* memory size n and the transition function δ_{mem}. At each step, the transition depends on the current state, the input on the tape and all the memory. The transition to the next step can update the entire memory. Formally: $\delta_{\text{mem}} : Q \times \Gamma \times \Gamma^n \to Q \times \Gamma \times \Gamma^n \times \{L, R\}$.

   Given a finite memory Turing Machine M_{mem}, define *formally* a (standard) Turing Machine M such that $L(M) = L(M_{\text{mem}})$. Namely, both Turing Machines accept the same language. Explain your answer.

   Bonus: Show more than one formal construction.

4. Let $L_1, L_2 \in RE \setminus R$ can the following occur? If your answer is “yes” give a concrete example. If your answer is “no” give a sketch of a proof.

   (a) $L_1 \cap L_2 \in R$

   (b) $L_1 \cup L_2 \in R$

   (c) $L_1 \cap L_2 \in R$ and $L_1 \cup L_2 \in R$

5. Prove or contradict: R is closed under infinite (countable) union. Namely, given $L_1, L_2 \ldots$ such that $L_i \in R$ then $\bigcup_{i=1}^{\infty} L_i \in R$.