Recitation 9 - Rice Theorem

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Rice Theorem

Definition

A property C is a subset of $\mathcal{RE}$ that complies to some conditions.

For example, $C = \{L \in RE \mid \epsilon \in L\}$.

Definition

A property C is not trivial if there exists two languages L_1, L_2 such that $L_1 \in C$ and $L_2 \notin C$.

Definition

Given a property C we define its language as:
$L_C = \{< M > \mid L(M) \in C\}$.

Rice Theorem (cont.)

Theorem

For every non trivial property C, $L_C \notin \mathcal{R}$.

Note: the properties are related to the language of the machine and not to the machine itself.

Rice Theorem - examples

Can we use Rice for the following languages?

Let $A_{TM_\epsilon} = \{< M > \ | \epsilon \in L(M)\}$
Yes! set $C = \{L \in \mathcal{RE} \mid \epsilon \in L\}$ and trivially $L_C = A_{TM_\epsilon}$.

C is not trivial as $\emptyset \notin C$ and $\Sigma^* \in C$.

Note that it is in $\mathcal{RE} \setminus \mathcal{R}$.
Simply build a TM M_{A_ϵ} such that given a TM M, runs M on ϵ and accepts if M accepts.

M_{A_ϵ} will be used in Slide 8.

Rice Theorem - examples

Can we use Rice for the following languages?

Let $A_{stops} = \{< M > \mid M \text{ always stops}\}$
No! This is a feature of the machine and not its language.

Rice Theorem - examples

Can we use Rice for the following languages?

Let $L_{\mathcal{RE}} = \{< M > \ | \ L(M) \in \mathcal{RE}\}$
No! the property $C = \{L \mid L \in \mathcal{RE}\}$ is trivial

(for every $L \in \mathcal{RE}$ it holds that $L \in C$)

Example 2

Prove $L_{\mathcal{R}} \notin \mathcal{R}$? where, $L_{\mathcal{R}} = \{< M > \ | \ L(M) \in \mathcal{R}\}$

Recall that $H_{TM} \in \mathcal{RE} \setminus \mathcal{R}$ where

$$H_{TM} = \{< M, w > \ | \ M \text{ is a TM that halts on } w\}$$

Define the property $C = \{L \mid L \in \mathcal{R}\}$.

C is not trivial as $H_{TM} \notin C$ and $\Sigma^* \in C$.

Thus, by Rice, $L_C = \{< M > | L(M) \in \mathcal{R}\} = L_{\mathcal{R}} \notin \mathcal{R}$

Example 2 (cont.)

Prove $L_{\mathcal{R}} \notin co-\mathcal{RE}$? where, $L_{\mathcal{R}} = \{< M > \ | \ L(M) \in \mathcal{R}\}$

Recall that $H_{TM} \in \mathcal{RE} \setminus \mathcal{R}$ where

$$H_{TM} = \{< M, w > \ | \ M \text{ is a TM that halts on } w\}$$

and that M_{A_ϵ} be a TM that given a TM M, runs M on ϵ and accepts if it accepts.

we will show a mapping reduction $H_{TM} \leq_M L_{\mathcal{R}}$:

We need a computable function $f(< M >, w) = M_w$

- M_w with input x runs M on w for $|x|$ steps.
- If M stops then M_w rejects
- If M does not stop then run M_{A_ϵ} on M and accept if M_{A_ϵ} accepts.

Example 2 (cont.)

$(< M >, w) \in H_{TM}$
$\Rightarrow M$ halts on w
$\Rightarrow \exists k$ s.t for every $|x| > k$, M_w rejects
$\Rightarrow L(M_w)$ is finite
$\Rightarrow M_w \in L_{\mathcal{R}}$

$(< M >, w) \notin H_{TM}$
$\Rightarrow M$ does not halt on w
$\Rightarrow L(M_w) = L(M_{A_\epsilon}) \notin \mathcal{R}$
$\Rightarrow M_w \notin L_{\mathcal{R}}$

Example 3

Prove $L_3 \notin \mathcal{RE} \cup co-\mathcal{RE}$? where, $L_3 = \{< M > \mid L(M) \text{is a CFL}\}$

Recall that $H_{TM} \in \mathcal{RE} \setminus \mathcal{R}$ where

$$H_{TM} = \{< M, w > \mid M \text{ is a TM that halts on } w\}$$

we will show two mapping reduction :
(i) $H_{TM} \leq_M L_3$ (This shows that $L_3 \notin co-\mathcal{RE}$)
(ii) $\overline{H_{TM}} \leq_M L_3$ (This shows that $L_3 \notin \mathcal{RE}$)

Example 3 - (i)

Let

$$M_{abc} = \text{A TM that decides } \{a^n b^n c^n \ s.t. \ n \in \mathbb{N}\}$$

We need a computable function $f(< M >, w) = M_w$

- M_w with input x runs M on w for $|x|$ steps.
- If M stops then M_w rejects
- If M does not stop then run M_{abc} on x and accept if M_{abc} accepts

$(< M >, w) \in H_{TM}$
$\Rightarrow M$ halts on w
$\Rightarrow \exists k$ s.t for every $|x| > k$, M_w rejects
$\Rightarrow L(M_w)$ is finite
$\Rightarrow M_w$ is a CFL

$(< M >, w) \notin H_{TM}$
$\Rightarrow M$ does not halt on w
$\Rightarrow L(M_w) = L(M_{abc})$ which is not a CFL

Example 3 - (ii)

Let

$$M_{abc} = \text{A TM that decides } \{a^n b^n c^n \text{ s.t. } n \in \mathbb{N}\}$$

We need a computable function $f(<M>, w) = M_w$

- M_w with input x runs M on w.
- If M stops then M_w runs M_{abc} on x and accept if M_{abc} accepts

$(<M>, w) \in \overline{H_{TM}}$
$\Rightarrow M$ does not halt on w
$\Rightarrow L(M_w) = \emptyset$
$\Rightarrow M_w$ is a CFL

$(<M>, w) \notin H_{TM}$
$\Rightarrow M$ halts on w
$\Rightarrow L(M_w) = L(M_{abc})$ which is not a CFL