Computational Models - Lecture 4[1]

Handout Mode

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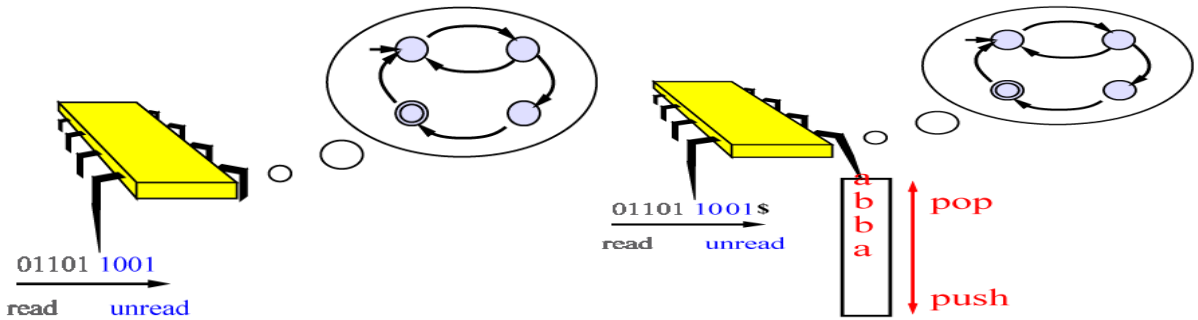
[1] Based on frames by Benny Chor, Tel Aviv University, modifying frames by Maurice Herlihy, Brown University.

Talk Outline

- Context Free Grammars/Languages (CFG/CFL)
- Algorithmic issues for CFL
- Pumping Lemma for context free languages

Next week:

- Push Down Automata (PDA)
- Equivalence of CFGs and PDAs



- Sipser's book, 2.1, 2.2 & 2.3

Short Overview of the Course

So far we saw

- finite automata,
- regular languages,
- regular expressions,
- Myhill-Nerode theorem
- pumping lemma for regular languages.

We now introduce stronger machines and languages with more expressive power:

- pushdown automata,
- context-free languages,
- context-free grammars,
- pumping lemma for context-free languages.

Context Free Grammars (CFG)

An example of a context free grammar, G_1:

- $\mathbf{A} \rightarrow 0A1$
- $A \rightarrow B$
- $B \rightarrow \#$

Terminology:

- Each line is a substitution rule or production.
- Each rule has the form: symbol $\rightarrow$ string.
  The left-hand symbol is a variable (usually upper-case).
- A string consists of variables and terminals.
- One variable is the start variable (lhs of top rule).

Rules for Generating Strings

- Write down the start variable.
- Pick a variable written down in current string and a derivation that starts with that variable.
- Replace that variable with right-hand side of that derivation.
- Repeat until no variables remain.
- Return final string (concatenation of terminals).

Process is inherently non deterministic.

Example

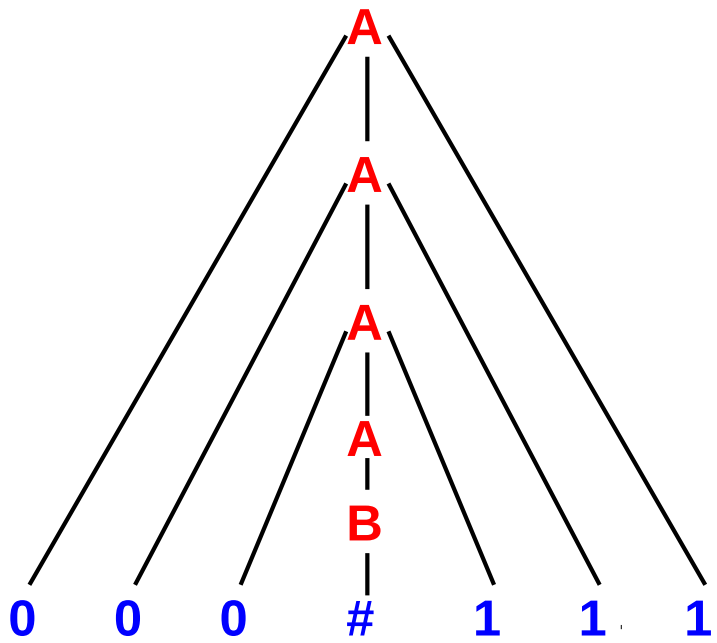
Grammar G_1:

- $A \to 0A1$
- $A \to B$
- $B \to \#$

Derivation with G_1:

$$\begin{aligned} \mathbf{A} &\to 0A1 \\ &\to 00A11 \\ &\to 000A111 \\ &\to 000B111 \\ &\to \mathbf{000\#111} \end{aligned}$$

Parsing Trees



Indifferent to derivation order.

Question 1

What strings can be generated in this way from the grammar G_1?

Answer: Exactly those of the form $0^n\#1^n$ ($n \geq 0$).

Context-Free Languages (CFL)

The language generated in this way is called the language of the grammar.

For example, $\mathcal{L}(G_1) = \{0^n \# 1^n : n \geq 0\}$.

Any language generated by a context-free grammar is called a context-free language.

A Useful Abbreviation

Rules with same variable on left hand side

$$A \rightarrow 0A1$$
$$A \rightarrow B$$

are written as:

$$A \rightarrow 0A1 \mid B$$

English-like Sentences

A grammar G_2 to describe a few English sentences:

$$
\begin{aligned}
< \text{SENTENCE} > &\rightarrow \mathcal{NP} < \text{VERB} > \\
\mathcal{NP} &\rightarrow < \text{ARTICLE} >< \text{NOUN} > \\
< \text{NOUN} > &\rightarrow \text{boy} \mid \text{girl} \mid \text{flower} \\
< \text{ARTICLE} > &\rightarrow \text{a} \mid \text{the} \\
< \text{VERB} > &\rightarrow \text{touches} \mid \text{likes} \mid \text{sees}
\end{aligned}
$$

Deriving English-like Sentences

A specific derivation in G_2:

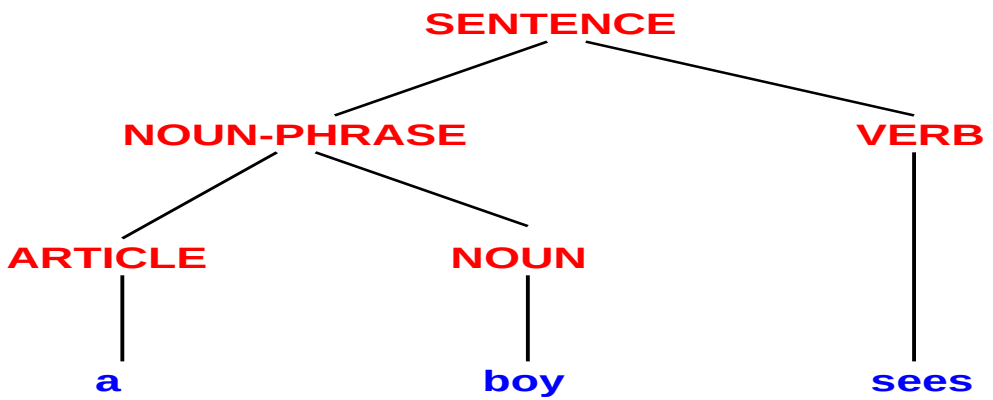
$$
\begin{aligned}
< \text{SENTENCE} > &\rightarrow \mathcal{NP} < \text{VERB} > \\
&\rightarrow < \text{ARTICLE} >< \text{NOUN} >< \text{VERB} > \\
&\rightarrow \text{a } < \text{NOUN} >< \text{VERB} > \\
&\rightarrow \text{a boy } < \text{VERB} > \\
&\rightarrow \text{a boy sees}
\end{aligned}
$$

More strings generated by G_2:

a flower sees
the girl touches

Derivation and Parse Tree

$$
\begin{aligned}
< \textbf{SENTENCE} > &\rightarrow \mathcal{NP} < \textbf{VERB} > \\
&\rightarrow < \textbf{ARTICLE} >< \textbf{NOUN} >< \textbf{VERB} > \\
&\rightarrow \textbf{a} < \textbf{NOUN} >< \textbf{VERB} > \\
&\rightarrow \textbf{a boy} < \textbf{VERB} > \\
&\rightarrow \textbf{a boy sees}
\end{aligned}
$$



Formal Definition

A context-free grammar is a 4-tuple (V, Σ, R, S), where

- V is a finite set of variables
- Σ is a finite set of terminals
- R is a finite set of rules: each rule is a variable and a finite string of variables and terminals.
- S is the start symbol.
- If u and v are strings of variables and terminals, and $A \to w$ is a rule of the grammar, then uAv yields uwv, written $uAv \to uwv$.
- $u \overset{*}{\to} v$ if $u = v$, or $u \to u_1 \to \ldots \to u_k \to v$ for some sequence $u_1, u_2, \ldots, u_k$

Definition 2

The language of the grammar G, denoted $\mathcal{L}(G)$, is $\{w \in \Sigma^* : S \overset{*}{\to} w\}$

where $\overset{*}{\to}$ is determined by G.

Example 1

$G_4 = (\{S\}, \{a, b\}, R, S)$.

R (Rules): $S \to aSb \mid SS \mid \varepsilon$

Some words in the language: $aabb$, $aababb$.

Question 3

What is this language?

Hint: Think of parentheses.

Example 2

$G_5 = (\{S\}, \{a, b\}, R, S)$.

R (Rules): $S \to aSa \mid bSb \mid \varepsilon$

Some words in the language: $abba$,$aabaabaa$.

Question 4

What is this language?

$\mathcal{L}(G_5) = \{ww^R : w \in \{a, b\}^*\}$

Proving $\mathcal{L}(G_5) = \{ww^R : w \in \{a,b\}^*\}$

What do we need to show?

1. If $z = ww^R$ then z is generated by G_5.
2. If z generated by G_5 then $z = ww^R$.

Part 1:
Proof by induction on the length of z.
Base: $|z| = 0$, then $S \to \varepsilon$. Hence $\varepsilon \in \mathcal{L}(G_5)$.
Inductive claim:
Let $|z| = 2k$.
Let $z = ww^R$ and $w = \sigma w'$. Then $z = \sigma w'(w')^R \sigma$.
Consider the derivation $S \to \sigma S \sigma$.
By the induction hypothesis $S \stackrel{*}{\to} w'(w')^R$, since $|w'(w')^R| = 2k - 2 < |z|$.
Therefore, G_5 generates $S \to \sigma S \sigma \stackrel{*}{\to} \sigma w'(w')^R \sigma = z$.
Hence $z \in \mathcal{L}(G_5)$.

Proving $\mathcal{L}(G_5) = \{ww^R : w \in \{a,b\}^*\}$

Part 2:
Assume that $S \stackrel{*}{\to} z$.
We prove by induction on the number of derivations that $z = ww^R$.
Base: If we have a single derivation, then the only possible derivation is $S \to \varepsilon$, and we are done.
Inductive claim:
Assume $S \stackrel{*}{\to} z$ in k derivations.
Assume we have $S \to \sigma S \sigma$ as the first derivation.
Then by the inductive hypothesis, $S \to w'(w')^R$.
Therefore, $z = \sigma w'(w')^R \sigma$.

Example 3

$G_6 = (\{S, A, B\}, \{a, b\}, R, S)$

R (Rules):

$S \to aB \mid bA \mid \varepsilon$

$A \to a \mid aS \mid bAA$

$B \to b \mid bS \mid aBB$

Some words in the language: $aababb$, $baabba$.

Question 5

What is this language?

$\mathcal{L}(G_6) = \{w \in \{a, b\}^* : \#_a(w) = \#_b(w)\}$

$\#_x(w)$ — number of occurrences of x in w

Proving $\mathcal{L}(G_6) = \{w \in \{a,b\}^* : \#_a(w) = \#_b(w)\}$

What do we need to show?

1. If w generated by G_6 then $\#_a(w) = \#_b(w)$.
2. If $\#_a(w) = \#_b(w)$ then w is generated by G_6.

Claim 6

If $S \stackrel{*}{\rightarrow} w$, then $\#_a(w) + \#_A(w) = \#_b(w) + \#_B(w)$.

Claim 7

For $w \in \{a,b\}^*$ let $k = k(w) = \#_a(w) - \#_b(w)$. Then:

1. If $k = 0$, then $S \stackrel{*}{\rightarrow} wS$
2. If $k > 0$, then $S \stackrel{*}{\rightarrow} wB^k$
3. If $k < 0$, then $S \stackrel{*}{\rightarrow} wA^k$

Are we done?!

How we prove these claims?

Proving Claim 7

Proof by induction on $|w|$:

- Basis: $w = \epsilon$ then $k(w) = 0$. Since $S \stackrel{*}{\to} S$, then $S \stackrel{*}{\to} \epsilon S$
- Induction step: for $w \in \{a, b\}^n$ write $w = w'\sigma$ with $|w'| = n - 1$

  Suppose $k' = k(w') = (\#_a(w') - \#_b(w')) > 0$ and $\sigma = a$

  By the induction hypothesis $S \stackrel{*}{\to} w'B^{k'}$

  Since $B \to aBB$, then $S \stackrel{*}{\to} wB^{k'+1} = wB^{k(w)}$

To complete the proof, need to show for $\sigma = b$ and for $k' \leq 0$

Designing Context-Free Grammars

No recipe in general, but few rules-of-thumb

- If CFG is the union of several CFGs, rename variables (not terminals) so they are disjoint, and add new rule $S \to S_1 \mid S_2 \mid \ldots \mid S_i$.
- For languages (like $\{0^n \# 1^n : n \geq 0\}$), with linked substrings, a rule of form $R \to uRv$ is helpful to force desired relation between substrings.
- For a regular language, grammar "follows" a DFA for the language (see next frame).

CFG for Regular Languages

Given a DFA : $M = (Q, \Sigma, \delta, q_0, F)$

CFG G for $\mathcal{L}(M)$:

1. Let R_0 be the staring variable
2. Add rule $R_i \to aR_j$, for any $q_i, q_j \in Q$ and $a \in \Sigma$ with $\delta(q_i, a) = q_j$
3. Add rule $R_i \to \varepsilon$, for any $q_i \in F$

Claim 8

$\mathcal{L}(G) = \mathcal{L}(M)$

Proof?

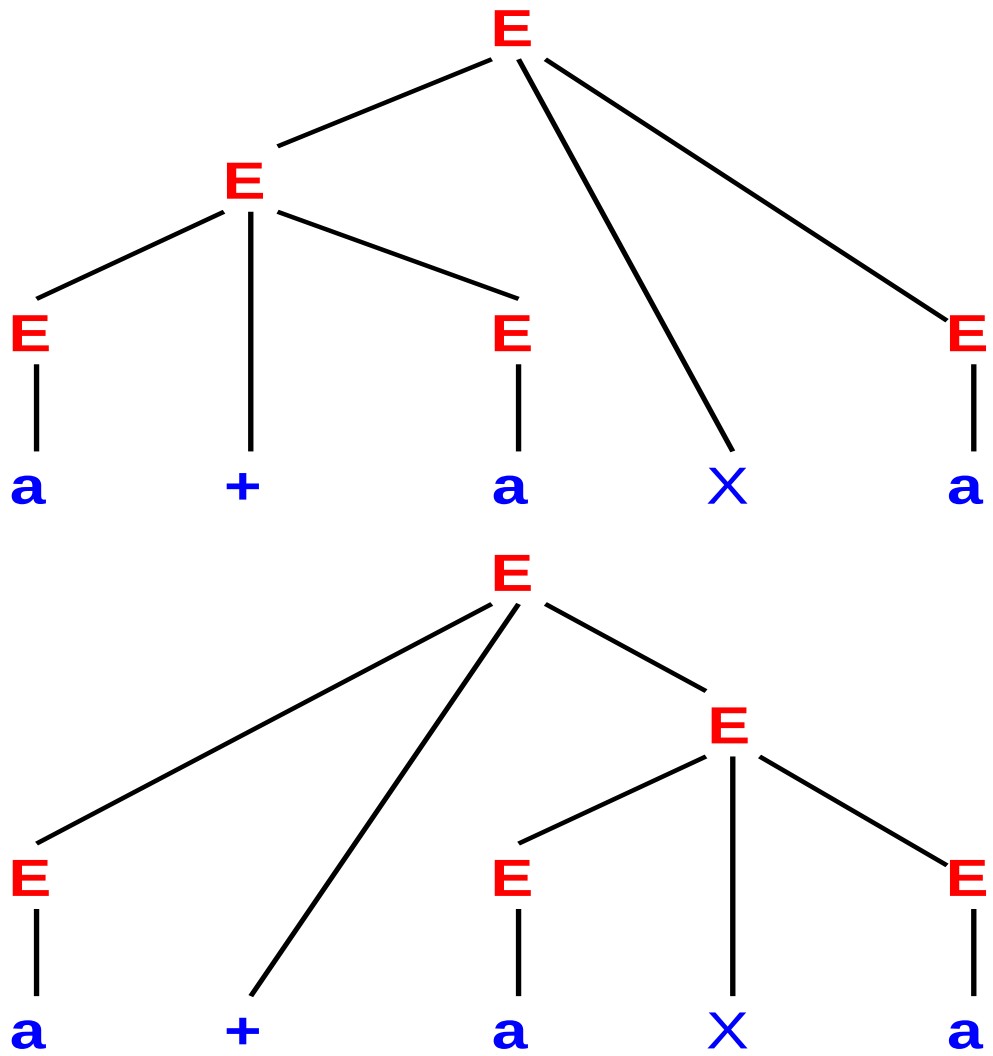
Claim 9

$R_0 \stackrel{*}{\to} wR_j$ iff $M(w) = q_j$.

Proof?Next class we'll see alternative proof via "Push-Down Automata"

Ambiguity in CFLs

Grammar G: $E \to E + E \mid E \times E \mid (E) \mid a$



Arithmetic Example

Consider the grammar $G' = (V, \Sigma, R, E)$, where

- $V = \{E, T, F\}$
- $\Sigma = \{a, +, \times, (,)\}$
- Rules:

$$\begin{aligned} E &\rightarrow E + T \mid T \\ T &\rightarrow T \times F \mid F \\ F &\rightarrow (E) \mid a \end{aligned}$$

Claim 10

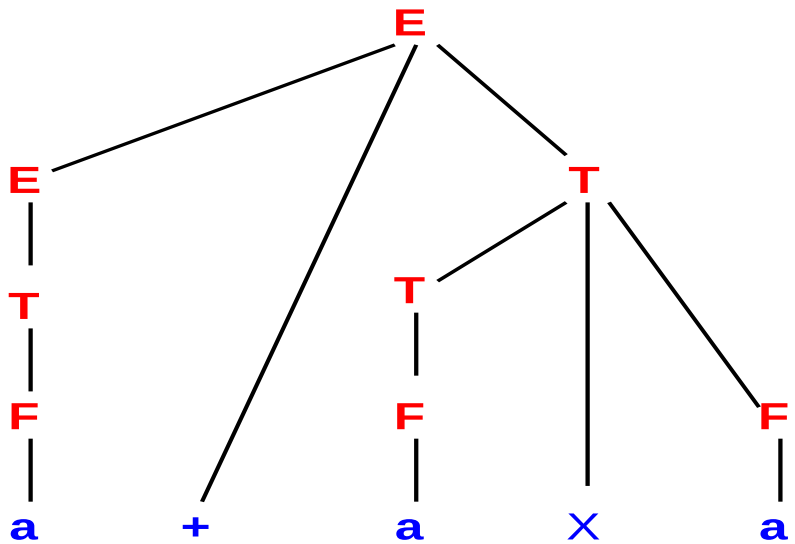
$\mathcal{L}(G') = \mathcal{L}(G)$

Proof?, but G' in not ambiguous.

Parsing Tree of G' for $a + a \times a$

G':

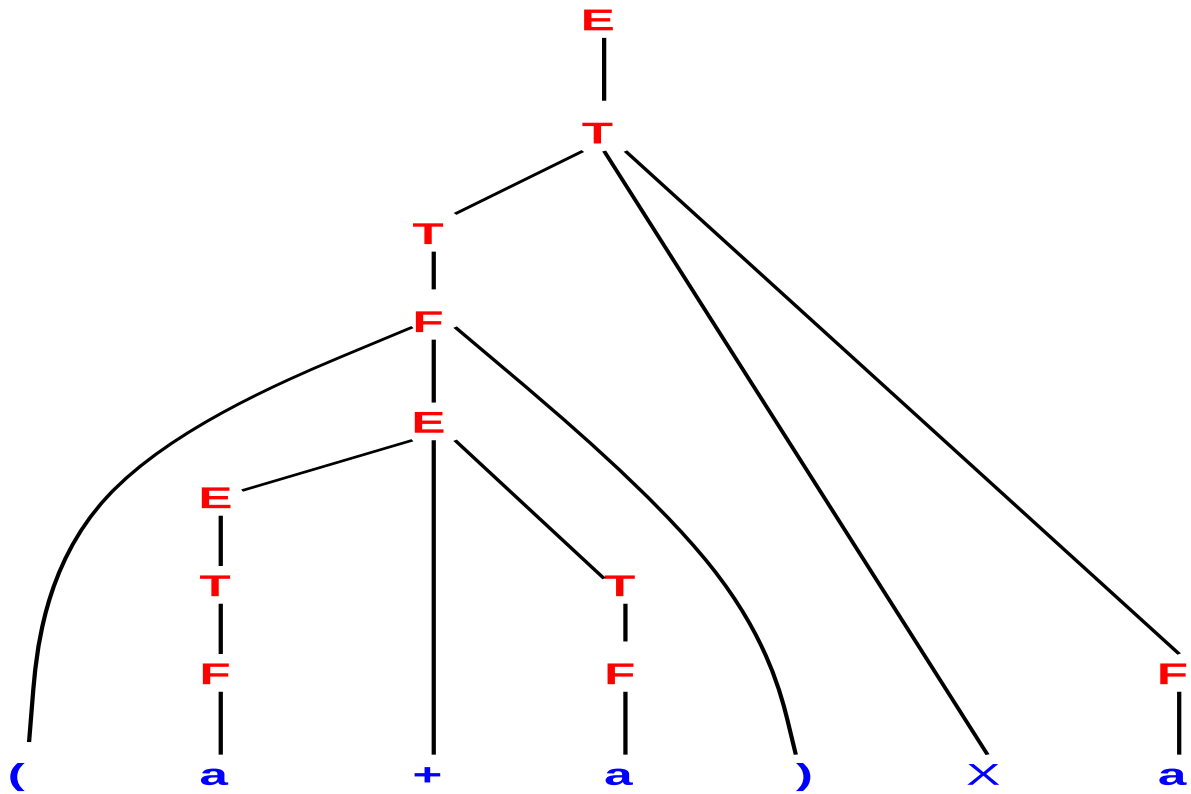
$$
\begin{aligned}
E &\rightarrow E + T \mid T \\
T &\rightarrow T \times F \mid F \\
F &\rightarrow (E) \mid a
\end{aligned}
$$



Parsing Tree of G' for $(a + a) \times a$

G':

$$
\begin{aligned}
E &\rightarrow E + T \mid T \\
T &\rightarrow T \times F \mid F \\
F &\rightarrow (E) \mid a
\end{aligned}
$$



Ambiguity

Definition 11

A string w is derived ambiguously from grammar G, if w has two or more different parse trees that generate it from G. A CFG is ambiguous, if it ambiguously derives a string.

- Ambiguity is usually not only a syntactic notion but also semantic, implying multiple meanings for the same string. Think of $a + a \times a$ from last grammar.
- It is sometime possible to eliminate ambiguity by finding a different context free grammar generating the same language. This is true for the arithmetic expressions grammar.

  Some languages are inherently ambiguous.

  Example: $\{1^i 2^j 3^k : \ i = j \vee j = k\}$

  Proof?

Part I

Checking Membership

Checking Membership in a CFL

Challenge

Given a CFG G and a string w, decide whether $w \in \mathcal{L}(G)$?

Initial Idea: Design an algorithm that tries all derivations.

Problem: If G does not generate w, we'll never stop.

Chomsky Normal Form (CNF)

A simplified, canonical form of context free grammars.
$G = (V, \Sigma, R, S)$ is in a CNF, if every rule in in R has one of the following forms:

$$\begin{array}{lll} A \to a, & A \in V & \wedge \quad a \in \Sigma \\ A \to BC, & A \in V & \wedge \quad B, C \in V \setminus \{S\} \\ S \to \varepsilon. & & \end{array}$$

Simpler to analyze: each derivation adds (at most) a single terminal, S only appears once, ε appears only at the empty word

CNF: Theorem

Theorem 12

Any context-free language is generated by a context-free grammar in Chomsky Normal Form.

Proof Idea:

- Add new start symbol S_0.
- Eliminate all ε rules of the form $A \to \varepsilon$.
- Eliminate all "unit" rules of the form $A \to B$.
- Patch up rules so that grammar generates the same language.
- Convert remaining "long rules" to proper form.

CNF: Example

$$
\begin{aligned}
S &\rightarrow ASA \mid aB \\
A &\rightarrow B \mid S \\
B &\rightarrow b \mid \varepsilon
\end{aligned}
$$

Is transformed into:

$$
\begin{aligned}
S_0 &\rightarrow AA_1 \mid UB \mid a \mid SA \mid AS \\
S &\rightarrow AA_1 \mid UB \mid a \mid SA \mid AS \\
A &\rightarrow b \mid AA_1 \mid UB \mid a \mid SA \mid AS \\
A_1 &\rightarrow SA \\
U &\rightarrow a \\
B &\rightarrow b
\end{aligned}
$$

CNF: Bounded Derivation Length

Lemma 13

For a CNF grammar G and $w \in \mathcal{L}(G)$ with $|w| = n \geq 1$, it holds that w has a derivation of length $2n - 1$.

Proof? consider the parsing tree for w

Advantage: Easier to check whether $w \in \mathcal{L}(G)$

Checking Membership for Grammars in CNF Form

Given a CNF grammar $G = (V, \Sigma, R, S)$, we build a function Derive(A, x) that returns TRUE iff $A \stackrel{*}{\to} x$.

Algorithm 14 (Derive(A, x))

- If $x = \varepsilon$: if $A \to \varepsilon \in R$ (i.e., $A = S$) return TRUE, otherwise return FALSE.
- If $|x| = 1$: if $A \to x \in R$ return TRUE, otherwise return FALSE.
- For each $A \to BC$ and each partition $x = x_1 x_2$:
  - Call Derive(B, x_1) and Derive(C, x_2).
  - Return TRUE if *both* return TRUE.
- Return FALSE.

Test whether $w \in \mathcal{L}(G)$ by calling Derive(S, w)

Correctness?

- Procedure Derive can also output a parse tree for w
- Have we critically used that G is in CNF?

Time Complexity of Derive

What is the time complexity $T : \mathbb{N} \mapsto \mathbb{N}$ of Derive?

- Each recursive call tests $|R|$ rules and n partitions.
  $T(n) \leq |R| \cdot n \cdot 2T(n-1)$
  $T(n) \in O((|R| \cdot n)^n)$.
- Still exponential...

Efficient Algorithm

- Keep in memory the results of Derive(A, x).
  - Number of different inputs: $|V| \cdot n^2$.
  - Only $|V| \cdot n^2$ calls, each takes $O(|R| \cdot n)$.
  - $T(n) \in O(|R| \cdot n^3 \cdot |V|)$.
- Polynomial time!

- This approach is called Dynamic Programming

  Basic idea:
  - If number of different inputs is limited, say $I(n)$.
  - Each run (excluding recursive calls) takes at most $R(n)$ time
  - Total running time is bounded by $T(n) \leq R(n)I(n)$.

Part II

Non-Context-Free Languages

Proving a Language is not a CFL

- The pumping lemma for finite automata and Myhill-Nerode theorem are our tools for showing that languages are not regular.
- We will now show a similar pumping lemma for context-free languages.
- It is slightly more complicated . . .

Pumping Lemma for CFL (also known as, the *uvxyz* Theorem)

Theorem 15

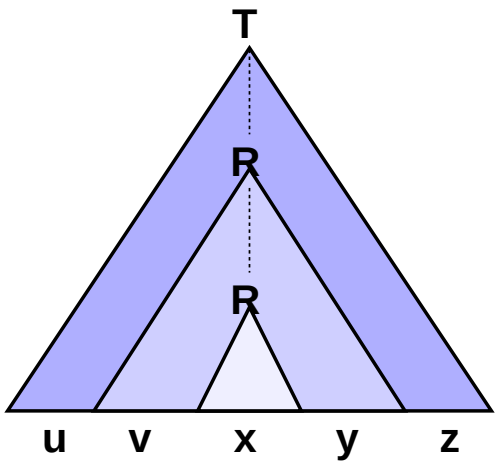
For any CFL $\mathcal{L}$ there exists $\ell \in \mathbb{N}$ ("critical length"), such that for any $w \in \mathcal{L}$ with $|w| \geq \ell$, there exit $u, v, x, y, z \in \Sigma^*$ such that $w = uvxyz$ and

- *For every $i \geq 0$: $uv^i xy^i z \in \mathcal{L}$*
- *$|vy| > 0$, ("non-triviality")*
- *$|vxy| \leq \ell$*

Basic Intuition:
Let $\mathcal{L}$ be a CFL and a let w be a "very long" string in $\mathcal{L}$. Then w must have a "tall" parse tree.

Hence, some root-to-leaf path must repeat a symbol. Why is that so?

Basic Intuition cont.



We have: $T \stackrel{*}{\rightarrow} uRz$, $R \stackrel{*}{\rightarrow} vRy$, and $R \stackrel{*}{\rightarrow} x$.

Proof of Thm 15

Let G be a CFG and let $\mathcal{L} = \mathcal{L}(G)$.

- Let b be the max number of symbols in right-hand-side of any rule (what is b for a CNF grammar?).

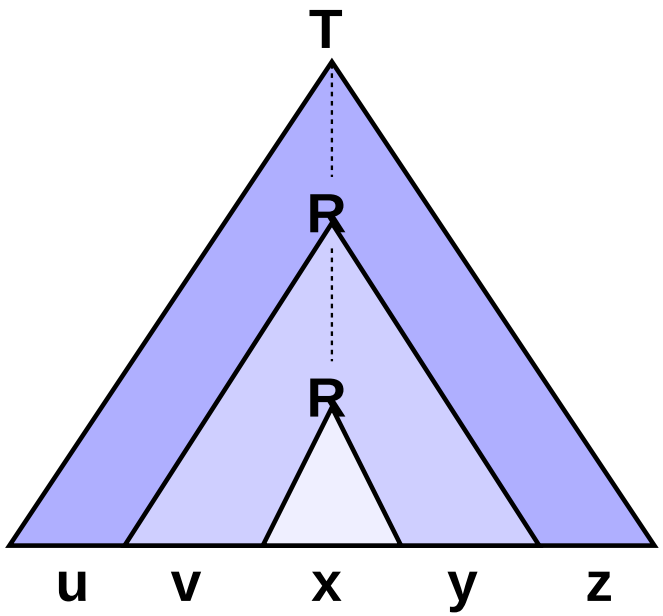
  Since no node in a parse tree of G has more than b children, at depth d such tree has at most b^d leaves.
- Let $|V|$ be the number of variables in G, and set $\ell = b^{|V|+2}$.

Let w be a string with $|w| \geq \ell$, and let T be parse tree for w (with respect to G) with fewest nodes

- T has height $\geq |V| + 2$
- Some path in T has length $\geq |V| + 2$
- Such path repeats a variable R

Proof of Thm 15 cont.

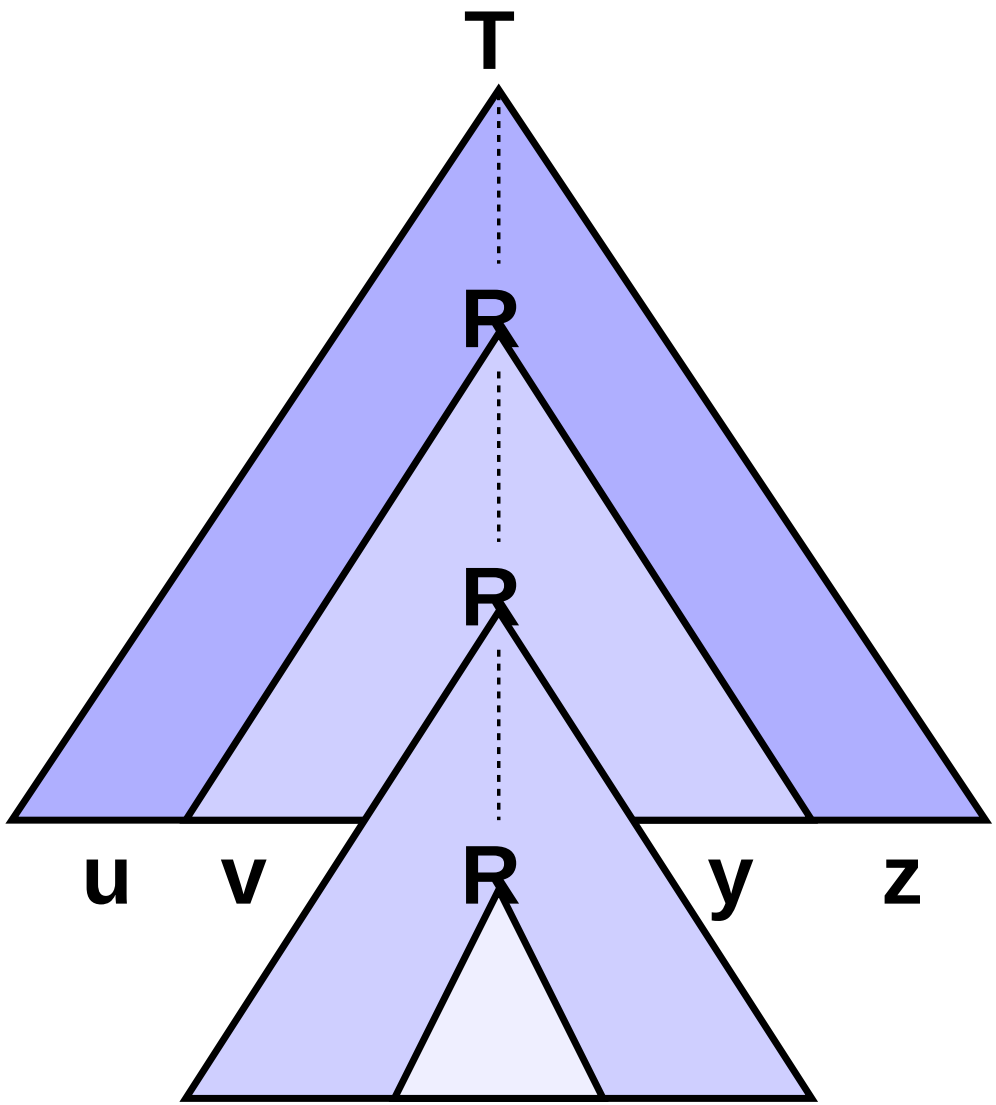
Set $w = uvxyz$



- Each occurrence of R produces a string
- Upper produces string vxy
- Lower produces string x

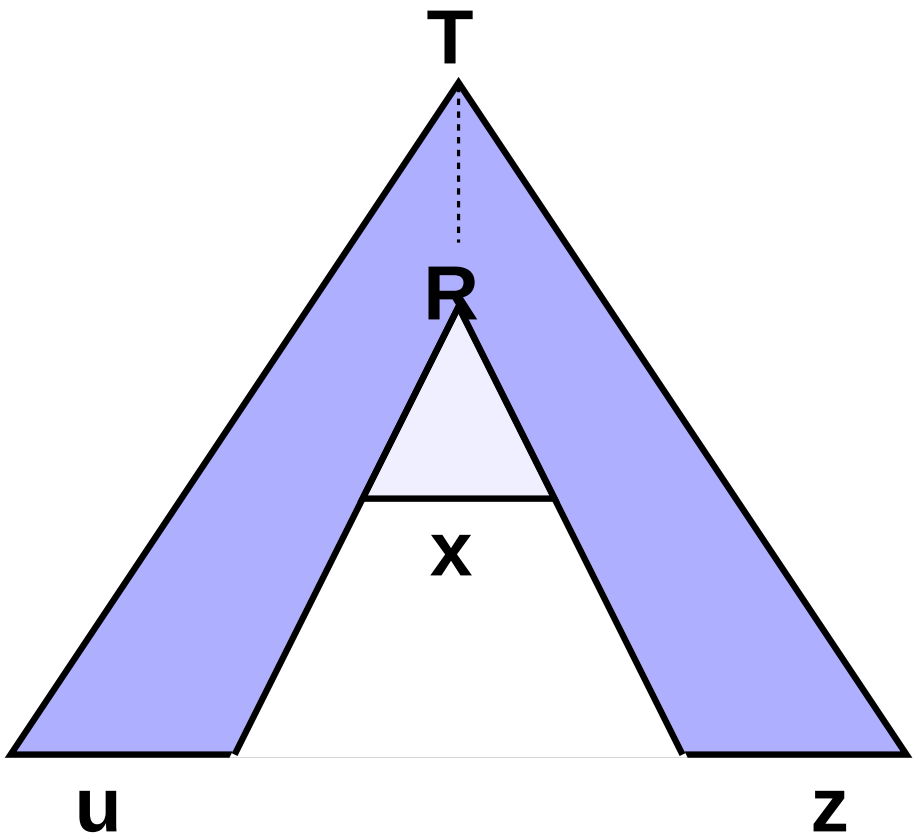
Proving $uv^ixy^iz \in \mathcal{L}$ for all $i > 1$

Replacing smaller by larger yields uv^ixy^iz, for $i > 0$.



Proving $uv^i xy^i z \in \mathcal{L}$ for $i = 0$

Replacing larger by smaller yields uxz.

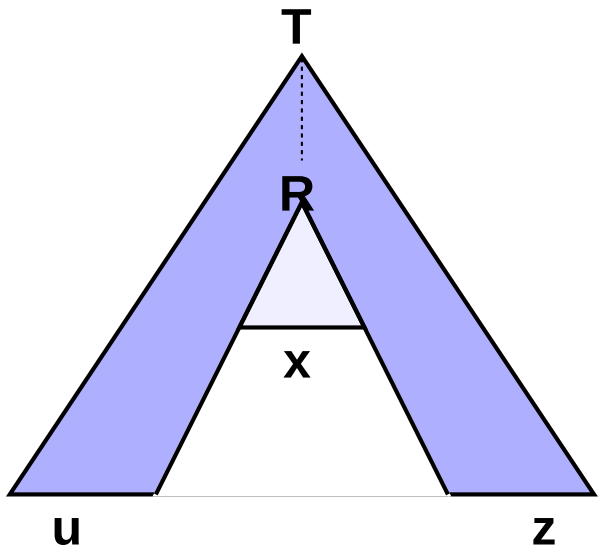


Together, they establish:

- $uv^i xy^i z \in \mathcal{L}$ for all $i \geq 0$

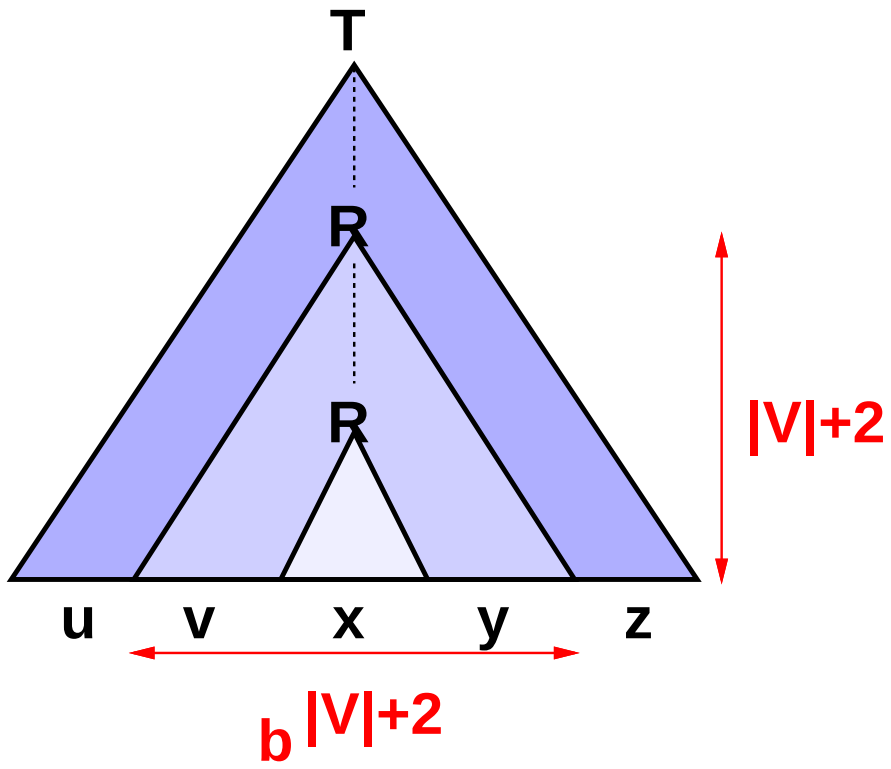
Proving $|vy| > 0$

If v and y are both ε, then



is a parse tree for w with fewer nodes than T, a contradiction.

Proving $|vxy| \leq \ell$



- Without loss of generality both occurrences of R lie in bottom $|V|+1$ variables on the path.
- The upper occurrence of R (from now on R^1) generates vxy.
- Subtree rooted at R^1 is of height at most $|V|+2$.
  Hence, $|vxy| \leq b^{|V|+2} = \ell$.

Non CFL Example (1)

Claim 16

$\mathcal{L}_1 = \{a^n b^n c^n : n \in \mathbb{N}\}$ is not a CFL.

Proof: By contradiction. Assume $\mathcal{L}_1$ is a CFL with grammar G, let ℓ be the critical length of G and consider $w = a^\ell b^\ell c^\ell$. Let u, v, x, y, z be the strings with $w = uvxyz$ guaranteed by Thm 15 for w.

- Note that neither v nor y contain
  - both a's and b's, or
  - both b's and c's,

  (otherwise uv^2xy^2z would have out-of-order symbols).
- But if v and y contain only one letter, then uv^2xy^2z is imbalanced

♣

Non CFL Example (2)

Claim 17

$\mathcal{L}_2 = \{a^i b^j c^k : 0 \leq i \leq j \leq k\}$ is not context free.

Proof: By contradiction. Assume $\mathcal{L}_2$ is a CFL with grammar G, let ℓ be the critical length of G and consider $w = a^\ell b^\ell c^\ell$. Let u, v, x, y, z be the strings with $w = uvxyz$ guaranteed by Thm 15 for w.

- Neither v nor y contains two distinct symbols, because otherwise uv^2xy^2z would have out-of-order symbols.
- vxy cannot be all the same letter (why?)
- $|vxy| \leq \ell$, so either
  - v contains only a's and y contains only b's, but then uv^2xy^2z has too few c's.
  - v contains only b's and y contains only c's, but then uv^0xy^0z has too many a's.

♣

Non CFL Example (3)

Claim 18

$\mathcal{L}_3 = \{ww : w \in \{0,1\}^*\}$ is not context-free.

Proof:
By contradiction. Assume $\mathcal{L}_3$ is a CFL with grammar G, let ℓ be the critical length of G and consider $w = 0^\ell 1^\ell 0^\ell 1^\ell$. Let u, v, x, y, z be the strings with $w = uvxyz$ guaranteed by Thm 15 for w.

- Assuming vxy is in the first half of w, then uv^2xy^2z "moves" a 1 into the first position of second half.
- Assuming vxy is in the second half, then uv^2xy^2z "moves" a 0 into the last position of first half.
- Assuming vxy straddles the midpoint, then pumping *down* to uxz yields $0^\ell 1^i 0^j 1^\ell$ where i and j cannot both be ℓ.

♣

Note that $\{ww^R : w \in \{0,1\}^*\}$ is a CFL.