Computational Models - Lecture 3[1]

Handout Mode

Iftach Haitner and Yishay Mansour.

Tel Aviv University.

March 3/5, 2014

[1]Based on frames by Benny Chor, Tel Aviv University, modifying frames by Maurice Herlihy, Brown University.

- The programs in the homework should be written in Python/Scheme.
- Non-regular languages: two approaches
  1. Pumping Lemma
  2. Myhill-Nerode Theorem (not in Sipser's book)
- Closure properties
- Algorithmic questions for NFAs

- Sipser, 1.4, 2.1, 2.2
- Hopcroft and Ullman, 3.4

Proved Last Time

Theorem 1

A language is described by a regular expression, iff it is regular.

We have made a lot of progress understanding what finite automata can do, but what they cannot do?

Negative Results

Is there a DFA that accepting

- $\mathcal{B} = \{0^n 1^n : n \geq 0\}$
- $\mathcal{C} = \{w : \#_1(w) = \#_0(w)\}$
- $\mathcal{D} = \{w : \#_{01}(w) = \#_{10}(w)\}$

$\#_s(w)$ – the number of times s appears in w.
All languages are over $\{0, 1\}$.

Consider $\mathcal{B}$:

- DFA must "remember" how many 0's it has seen
- Impossible with finite state.

The others languages seem to be exactly the same...

Question: Is this a proof?

Answer: No, $\mathcal{D}$ is regular.....

Part I

Pumping Lemma

Regular languages can be pumped

For any regular language $\mathcal{L}$ there exists $\ell > 0$ (the pumping length) s.t.:
Any $s \in \mathcal{L}$ longer than ℓ, can be "pumped" into a longer string in $\mathcal{L}$.

This is a powerful technique for showing that a language is not regular.

The Pumping Lemma

Lemma 2

For any regular language $\mathcal{L}$, exists $\ell > 0$ (the pumping length) s.t.: every $s \in \mathcal{L}$ with $|s| \geq \ell$ can be written as $s = xyz$ such that:

1. $xy^iz \in \mathcal{L}$ *for every* $i \geq 0$,
2. $|y| > 0$, *and*
3. $|xy| \leq \ell$.

Remarks: Without the second condition, the theorem would be trivial.

The third condition is technical and sometimes useful.

Proving the Pumping Lemma

Let $M = (Q, \Sigma, \delta, q_1, F)$ be a DFA accepting $\mathcal{L}$, and let $\ell = |Q|$.

Let $s \in \mathcal{L}$ be with $|s| \geq \ell$, and consider the sequence of states M traverse as it reads $s = s_1 \ldots s_n$:

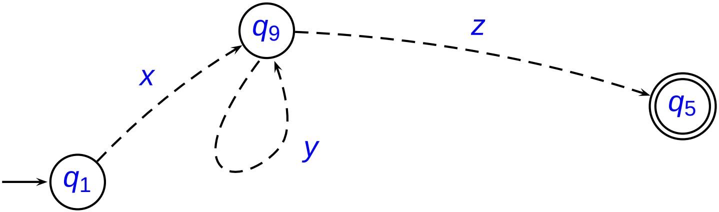
$$\begin{array}{ccccccccccccccccc}
 & s_1 & & s_2 & & s_3 & & s_4 & & s_5 & & s_6 & & \ldots & & s_n & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
q_1 & & q_{20} & & q_9 & & q_{17} & & q_{12} & & q_{13} & & q_9 & & q_2 & & q_5 \in F
\end{array}$$

By the pigeonhole principle, at least one of the states in the above sequence repeats. (?)

Proving the Pumping Lemma, cont.

Let q_9 be the repeating state.

Write $s = xyz$



- By inspection, M accepts $xy^k z$ for *every* $k \geq 0$.
- $|y| > 0$, because the state q_9 is repeated.
- To ensure that $|xy| \leq \ell$, pick first state repetition, which must occur no later than $\ell + 1$ states in sequence.

Application # 1

Corollary 3

$\mathcal{B} = \{0^n 1^n : n > 0\}$ is not regular.

Proof: By contradiction. Suppose $\mathcal{B}$ is regular and let ℓ be its pumping length.

- Consider the string $s = 0^\ell 1^\ell \in \mathcal{B}$.
- Let x, y, z be (one possible) strings guaranteed by the pumping lemma (i.e., $s = xyz$)
  1. $xy^i z \in \mathcal{B}$ for *every* $k \geq 0$,
  2. $|y| > 0$, and
  3. $|xy| \leq \ell$.
- If y is all 0, then xy^2z has too many 0's.
- If y is all 1, then xy^2z has too many 1's.
- If y is mixed, then xy^2z is not of right form. ♣

We did not use the third property.

Application # 2

Corollary 4

$\mathcal{C} = \{w : \#_1(w) = \#_0(w)\}$ *is not regular.*

Proof: By contradiction. Suppose $\mathcal{C}$ is regular. Let ℓ be the pumping length.

- Consider the string $s = 0^\ell 1^\ell \in \mathcal{C}$.
- Let x, y, z be (one possible) strings guaranteed by the pumping lemma (i.e., $s = xyz$)
  1. $xy^iz \in \mathcal{B}$ for *every* $k \geq 0$,
  2. $|y| > 0$, and
  3. $|xy| \leq \ell$.
- Since $|xy| \leq \ell$, the string y is all 0's.
- Thus, $xy^2z \notin \mathcal{C}$ (more 0's than 1's). ♣

Could we have used $s = (01)^\ell$?

Application # 3

Corollary 5

$\mathcal{E} = \{0^i 1^j : i > j\}$ is not regular.

Proof: By contradiction. Suppose $\mathcal{E}$ is regular. Let ℓ be its pumping length.

- Consider the string $s = 0^\ell 1^{\ell-1} \in \mathcal{E}$.
- By pumping lemma, $s = xyz$, where $xy^k z \in \mathcal{E}$ for every $k \geq 0$, $|y| > 0$ and $|xy| \leq \ell$.
- But $xy^0 z = xz \notin \mathcal{E}$ (at least as much 1's as 0's) ♣

Application # 4

Corollary 6

The language Primes $\subset \{0,1\}^*$ *– all strings whose length is a prime number – is not regular.*

Proof: Suppose *Primes* is regular accepted by DFA M, and let ℓ be its pumping length.

- Let $s = 1^p \in Primes$, where $p \geq \ell$ is a prime (?)
- By pumping lemma, $s = xyz$, where $xy^kz \in Primes$ for every $k \geq 0$.
- Let $|y| = m$. Hence, $xy^{p+1}z = 1^{p+mp} \in Primes$
- but $p(m+1)$ is not prime... ♣

Another Example

Consider the language $\mathcal{L} = \{a^i b^n c^n : n \geq 0, i \geq 1\} \cup \{b^n c^m : n, m \geq 0\}$.

Any $s \in \mathcal{L}$ can be pumped:

- If $s = a^i b^n c^n$, then set $x = \varepsilon$ and $y = a$.
- If $s = b^n c^m$, then set $x = \varepsilon$ and $y = b$.
- If $s = c^m$, then set $x = \varepsilon$ and $y = c$.

  (in all cases z is set arbitrarily).

- Is $\mathcal{L}$ regular? No
- How can we prove it?

Part II

Characterization of Regular Languages

The equivalence relation $\overset{\mathcal{L}}{\sim}$

Definition 7

For $\mathcal{L} \subseteq \Sigma^*$, define the equivalence relation $\overset{\mathcal{L}}{\sim}$ over words in Σ^*, by $x \overset{\mathcal{L}}{\sim} y$ if for every $z \in \Sigma^*$, it holds that $xz \in \mathcal{L} \iff yz \in \mathcal{L}$.

It is easy to see that $\overset{\mathcal{L}}{\sim}$ is indeed an equivalence relation (reflexive, symmetric, transitive) on Σ^*.
Hence, $\overset{\mathcal{L}}{\sim}$ partitions Σ^* into equivalence classes.

For $x \in \Sigma^*$, let $[x] \subseteq \Sigma^*$ denote its equivalence class with respect to $\overset{\mathcal{L}}{\sim}$

How many equivalence classes does $\overset{\mathcal{L}}{\sim}$ induce? finite or infinite?

Could be either (depends on $\mathcal{L}$).

Fact 8 (right invariance)

If $x \overset{\mathcal{L}}{\sim} y$, then $xw \overset{\mathcal{L}}{\sim} yw$ for every $w \in \Sigma^*$

Three Examples

- $\mathcal{L}_1 = \{w : \#_1(w) \bmod 4 = 0\}$

  $\overset{\mathcal{L}_1}{\sim}$ has finitely many equivalence classes.

  The equivalent classes are: [1], [11], [111], [1111]
- $\mathcal{L}_2 = \{0^n 1^n : n \in \mathbb{N}\}$

  $\overset{\mathcal{L}_2}{\sim}$ has infinitely many equivalence classes.

  $[0] \neq [0^2] \neq [0^3] \ldots$
- $\mathcal{L}_3 = \{a^i b^n c^n : n \geq 0, i \geq 1\} \cup \{b^n c^m : n, m \geq 0\}$

  $\overset{\mathcal{L}_3}{\sim}$ has infinitely many equivalence classes.

  $[ab] \neq [ab^2] \neq [ab^3] \neq \ldots$

The above statements required a proof...

Myhill-Nerode Theorem

Theorem 9 (Myhill-Nerode Theorem)

$\mathcal{L} \subseteq \Sigma^*$ is regular iff $\overset{\mathcal{L}}{\sim}$ finitely many equivalence classes.

Hence

- $\mathcal{L}_1 = \{w \in \{0,1\}^* : \#_1(w) \bmod 4 = 0\}$ is regular.
- $\mathcal{L}_2 = \{0^n 1^n : n \in \mathbb{N}\}$ is not regular.
- $\mathcal{L}_3 = \{a^i b^n c^n : n \geq 0, i \geq 1\} \cup \{b^n c^m : n, m \geq 0\}$ is not regular.

Proving Myhill-Nerode Theorem $\Longrightarrow$

Let $\mathcal{L}$ be a regular language and let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA accepting it.

- Define the binary relation $\overset{M}{\sim}$ by $x \overset{M}{\sim} y$ if $\widehat{\delta}(q_0, x) = \widehat{\delta}(q_0, y)$.
- $\overset{M}{\sim}$ is an equivalence relation.
- $x \overset{M}{\sim} y \implies xz \overset{M}{\sim} yz$ for every $z \in \Sigma^*$.

  $\implies xz \in \mathcal{L}$ iff $yz \in \mathcal{L}$.
- Hence, $x \overset{M}{\sim} y \Longrightarrow x \overset{\mathcal{L}}{\sim} y$.
- Each equivalence class of $\overset{\mathcal{L}}{\sim}$ corresponds to union of classes of $\overset{M}{\sim}$. Namely, $\overset{M}{\sim}$ is a refinement of $\overset{\mathcal{L}}{\sim}$. (see drawing on board)
- Specifically, $\#$ of equivalence classes of $\overset{\mathcal{L}}{\sim}$ is less or equal than $\#$ of equivalence classes of $\overset{M}{\sim}$.
- $\overset{M}{\sim}$ has finitely many equivalence classes. (?)
- Therefore, $\overset{\mathcal{L}}{\sim}$ has finitely many equivalence classes. ♠

Proving Myhill-Nerode Theorem $\Longleftarrow$

Assume $\overset{\mathcal{L}}{\sim}$ has finitely many equivalence classes and let $x_1, \ldots, x_n \in \Sigma^*$ be their representatives.
We'll construct a DFA $M = (Q, \Sigma, \delta, q_0, F)$ that accepts $\mathcal{L}$.
For $x \in \Sigma^*$, let $C(x)$ be the index $i \in \{1, \ldots, n\}$ with $x \in [x_i]$.

- $Q = \{1, \ldots, n\}$.
- $\delta(i, a) = C(x_i a)$.
- $q_0 = C(\varepsilon)$.
- $F = \{i \colon x_i \in \mathcal{L}\}$.

Claim. Let $x \in [x_i]$, then $\widehat{\delta}(q_0, x) = i$.

Proof: By induction on word length.

1. Assume $x \in [x_i]$ and $xa \in [x_j]$.
2. By right invariance, $\delta(i, a) = j$.
3. By i.h., $\widehat{\delta}(q_0, xa) = \delta(\widehat{\delta}(q_0, x), a) = \delta(i, a) = j$.

Therefore, M accepts x iff $x \in \mathcal{L}$. ♠.

This is the optimal DFA, number of states wise, for $\mathcal{L}$.

Example

Construct a DFA for $\{w : \#_1(w) \bmod 5 = 0\}$, via the latter proof method.

Finding the minimal automata

Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$, find the minimal (with respect to $\#$ of states) DFA M' with $\mathcal{L}(M') = \mathcal{L}(M)$.

States $q_1, q_2 \in Q$ are equivalent, if for all $x_1, x_2 \in \Sigma^*$ with $\widehat{\delta}(q_0, x_i) = q_i$, it holds that $x_1 \overset{\mathcal{L}}{\sim} x_2$.

Idea: keep merging equivalent states in Q, until all states are non-equivalent.

Actual idea:

1. Start with the two sets F and $Q \setminus F$.
2. Keep *splitting* the sets until all states in the same set are equivalent.

   To check whether states q and q' are equivalent, check if $\delta(q, a)$ and $\delta(q', a)$ are in the same set, for all $a \in \Sigma$.
3. Merge all states in the same set.

We assume for simplicity that M has no unreachable states(?)

Finding the minimal automata

Algorithm 10

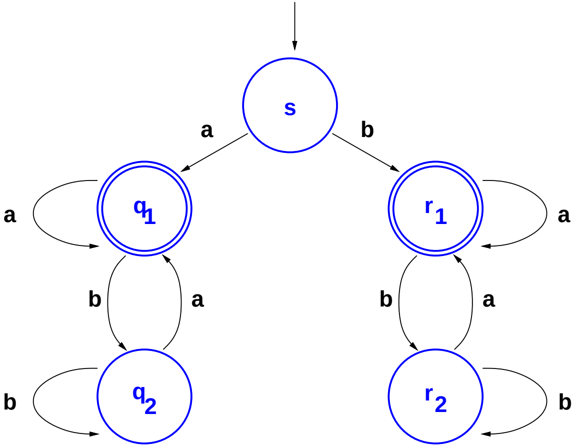
Input: DFA $M = (Q, \Sigma, \delta, q_0, F)$

1. Let $\mathcal{T} = \{F, Q \setminus F\}$.
2. While $\exists\, \mathcal{S} \in \mathcal{T}$, $q_1, q_2 \in \mathcal{S}$ and $\sigma \in \Sigma^*$ s.t, $\delta(q_1, \sigma) \in \mathcal{S}'$ and $\delta(q_2, \sigma) \notin \mathcal{S}'$ for some $\mathcal{S}' \in \mathcal{T}$:
   1. Let $\mathcal{S}_{sp} = \{q \in \mathcal{S} : \delta(q, \sigma) \in \mathcal{S}'\}$.
   2. Set $\mathcal{T} = \mathcal{T} \cup \mathcal{S}_{sp} \cup (\mathcal{S} \setminus \mathcal{S}_{sp}) \setminus \mathcal{S}$.
3. Output DFA $M' = (Q', \delta', q'_0, F')$, where
   - $Q' = \mathcal{T}$
   - $q'_0 = \mathcal{S}_0 \in \mathcal{T}$, where $q_0 \in \mathcal{S}_0$.
   - $F' = \{\mathcal{S} \in \mathcal{T} : \mathcal{S} \subseteq F\}$
   - $\delta'(\mathcal{S}, \sigma) = \mathcal{S}' \in \mathcal{T}$, s.t. $\delta(q, \sigma) \in \mathcal{S}'$ for any $q \in \mathcal{S}$.

Claim 11

The above algorithm outputs the minimal automata for $\mathcal{L}(M)$.

Example



Part III

Closure Properties of Regular Languages

Simple Closure Properties

- Regular languages are closed under complement.
  1. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA that accepts $\mathcal{L}$.
  2. Then $M' = (Q, \Sigma, \delta, q_0, Q \setminus F)$ is a DFA that accepts $\overline{\mathcal{L}} = \Sigma^* \setminus \mathcal{L}$.
  3. NFA ?!
- Regular languages are closed under intersection.
  1. $\mathcal{L}_1 \cap \mathcal{L}_2 = \overline{\overline{\mathcal{L}_1} \cup \overline{\mathcal{L}_2}}$.
  2. Proof with automata ?

Division

For languages $\mathcal{L}_1, \mathcal{L}_2 \in \Sigma^*$, define

$$\mathcal{L}_1/\mathcal{L}_2 = \{x \in \Sigma^* \colon \exists y \in \mathcal{L}_2,\ xy \in \mathcal{L}_1\}$$

Examples:

- $\mathcal{L}_1 = (01 \cup 1)^*$ and $\mathcal{L}_2 = 00$. Then $\mathcal{L}_1/\mathcal{L}_2 = \emptyset$
- $\mathcal{L}_3 = a^*b^*c^*$ and $\mathcal{L}_4 = b$. Then $\mathcal{L}_3/\mathcal{L}_4 = a^*b^*$

Closure under division

Recall, $\mathcal{L}_1/\mathcal{L}_2 = \{x : \exists y \in \mathcal{L}_2, xy \in \mathcal{L}_1\}$

Theorem 12

Regular languages are closed under division with any language.

Proof: Let $\mathcal{L}_1$ be a regular language and let $\mathcal{L}_2$ be an arbitrary language.

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA for $\mathcal{L}_1$.
- Let $F' = \{q \in Q : \exists y \in \mathcal{L}_2, \delta(q, y) \in F\}$
- The DFA $M' = (Q, \Sigma, \delta, q_0, F')$ accepts $\mathcal{L}_1/\mathcal{L}_2$. ♣

F' is well defined, but might be hard to compute – “non constructive proof".

Homomorphism

Definition 13 (Homomorphism)

An homomorphism from alphabet Δ to words over alphabet Σ, is a function $h\colon \Delta \mapsto \Sigma^*$.
For $w \in \Delta^*$, let $h(w = w_1, \ldots, w_n) = h(w_1) \cdots h(w_n)$.
For $\mathcal{L} \subseteq \Delta^*$, let $h(\mathcal{L}) = \{h(w)\colon w \in \mathcal{L}\}$.

Examples:

- Let $h\colon \{0,1\} \mapsto \{a,b\}^*$ be defined by $h(1) = aba$ and $h(0) = aa$. $h(010) = aa\,aba\,aa$. For $\mathcal{L}_1 = (01)^*$, $h(\mathcal{L}_1) = (aaaba)^*$.
- Let $h(0) = a$, $h(1) = a$. For $\mathcal{L}_2 = \{0^n 1^n\colon n \geq 0\}$, $h(\mathcal{L}_2) = \{a^{2n}\colon n \geq 0\}$.

Theorem 14

Regular languages are closed under homomorphism.

Proof: two options:

- Using regular expressions
- Using Automata

Inverse homomorphism

Definition 15 (Inverse homomorphism)

For homomorphism $h\colon \Delta \mapsto \Sigma^*$, define its inverse homomorphism $h^{-1}\colon \Sigma^* \mapsto P(\Delta^*)$, by $h^{-1}(w) = \{x \in \Delta^* \colon h(x) = w\}$.

For $\mathcal{L} \subseteq \Sigma^*$, let $h^{-1}(\mathcal{L}) = \bigcup_{x \in \mathcal{L}} h^{-1}(x) = \{x \in \Delta^* \colon h(x) \in \mathcal{L}\}$

Example: $h(1) = aba$, $h(0) = aa$ and $\mathcal{L}_2 = (ab \cup ba)^* a$.
Then $h^{-1}(\mathcal{L}_2) = \{1\}$. ($\mathcal{L}_2$ has no words starting with $h(0)$ or $h(1\sigma)$).

Claim 16

For any $h\colon \Delta \mapsto \Sigma^*$:

1. $h(h^{-1}(\mathcal{L})) \subseteq \mathcal{L}$, for any $\mathcal{L} \subseteq \Sigma^*$
2. $\mathcal{L} \subseteq h^{-1}(h(\mathcal{L}))$, for any $\mathcal{L} \subseteq \Delta^*$

Proof:

1. Immediate
2. Holds since $w \in h^{-1}(h(w))$ for any $w \in \Delta^*$

Closure under inverse homomorphism

Theorem 17

Regular languages are closed under inverse homomorphism.

Proof idea: Let $\mathcal{L}$ be a regular language, let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA for $\mathcal{L}$ and let $h \colon \Delta \mapsto \Sigma^*$.

- For each $a \in \Delta$, we advance in M using $h(a)$.
- Formally, we define $M' = (Q, \Delta, \delta', q_0, F)$, where $\delta'(q, a) = \widehat{\delta}(q, h(a))$.
- Hence, $\widehat{\delta'}(q, w) = \widehat{\delta}(q, h(w))$
- $w \in \mathcal{L}(M') \longleftrightarrow h(w) \in \mathcal{L}(M)$ ♣

Using Homomorphism

We know that $\mathcal{L}_1 = \{0^n 1^n : n \geq 1\}$ is not regular, show that $\mathcal{L}_2 = \{a^n b a^n : n \geq 1\}$ is not regular.

We will prove using homomorphism and inverse homomorphism. Let

- $h_1(a) = a$, $h_1(b) = b$, $h_1(c) = a$. $(h_1 : \{a, b, c\} \mapsto \{a, b, c\}^*)$
- $h_2(a) = 0$, $h_2(b) = \epsilon$, $h_2(c) = 1$. $(h_1 : \{a, b, c\} \mapsto \{0, 1\}^*)$

We prove $h_2(h_1^{-1}(\mathcal{L}_2) \cap a^*b^*c^*) = \mathcal{L}_1$. Thus, $\mathcal{L}_2$ is not regular (?)

- $h_1^{-1}(\mathcal{L}_2) = (a \cup c)^n b (a \cup c)^n$
- $h_1^{-1}(\mathcal{L}_2) \cap a^*b^*c^* = \{a^n b c^n : n \geq 1\}$
- $h_2(h_1^{-1}(\mathcal{L}_2) \cap a^*b^*c^*) = \{0^n 1^n : n \geq 1\}$

Part IV

Algorithmic Questions for NFAs

Algorithmic Questions for NFAs

Q.: Given an NFA, N, and a string w, is $w \in \mathcal{L}(N)$?

Answer: Construct the DFA equivalent to N and run it on w.

Q.: Is $\mathcal{L}(N) = \emptyset$?
Answer: This is a reachability question in graphs: Is there a path in the states' graph of N from the start state to some accepting state?

There are simple, efficient algorithms for this task.

More Questions

Q.: Is $\mathcal{L}(N) = \Sigma^*$?

Answer: Check if $\overline{\mathcal{L}(N)} = \emptyset$.

Q.: Given N_1 and N_2, is $\mathcal{L}(N_1) \subseteq \mathcal{L}(N_2)$?

Answer: Check if $\overline{\mathcal{L}(N_2)} \cap \mathcal{L}(N_1) = \emptyset$.

Q.: Given N_1 and N_2, is $\mathcal{L}(N_1) = \mathcal{L}(N_2)$?

Answer: Check if $\mathcal{L}(N_1) \subseteq \mathcal{L}(N_2)$ and $\mathcal{L}(N_2) \subseteq \mathcal{L}(N_1)$.

In the future, we will see that for stronger models of computations, many of these problems cannot be solved by any algorithm.

Part V

Summary — Regular Languages

Summary - Regular Languages

So far we saw

- Finite automata,
- Regular languages,
- Regular expressions,
- Myhill-Nerode theorem and pumping lemma for regular languages.

Next class we introduce stronger machines and languages with more expressive power:

- pushdown automata,
- context-free languages,
- context-free grammars